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EXISTENCE AND STABILITY RESULTS FOR DIFFERENTIAL EQUATIONS WITH COMPLEX ORDER INVOLVING HILFER FRACTIONAL DERIVATIVE

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Abstract. In this paper, we study dynamical behaviour of differential equation involving Hilfer fractional derivative with complex order. The existence of solution, Ulam-Hyers stability and generalized Ulam-Hyers-Rassias stability results are investigated.

Keywords: Hilfer fractional derivative, existence, Ulam-Hyers stability, complex order.

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1. Introduction

Fractional calculus deals with the study of fractional order integral and derivative operators over real or complex domains and their applications are in the area of fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc. There has been significant development in fractional differential equations in recent years; see the monographs of Kilbas et al. [8], Podlubny [13], Lakshmikantham et al. [10], Tomovski et al. [18]. The most common used fractional derivatives are Riemann and Liouville, Caputo. The investigation of these derivatives has been done already, readers can refer to [3, 9, 10, 16, 17, 20]. The generalisation of Riemann-Liouville and Caputo derivatives was initiated by Hilfer [4]. Recently many researchers focused their interest on Hilfer fractional derivative, see [2, 4, 6, 7, 15]. Love [11] developed sufficient conditions for existence of derivatives of imaginary order, and its scope. Abdolali Neamaty et al. [12] considered the fractional boundary value problem for differential equation of complex-order. Most recently, Atanackovic et al. [1] introduced complex order fractional derivatives in models that describe viscoelastic materials. Further D. Vivek et al. [19] analyzed Ulam stability results for integro-differential equations with complex order. In particular, the Ulam- Hyers stability and Ulam-Hyers-Rassias stability have been taken up by number of mathematicians and the study of this area has the grown to be one of the central subjects in the mathematical analysis area. For more details reader can refer to [5, 14, 19]. Inspired by the above discussion, we introduce complex order to Hilfer fractional derivative.

Consider the Hilfer fractional derivative of complex order is as follows

$$D^{\theta_1,\theta_2}x(t) = f(t, x(t)),\ t \in J := [0, T], \tag{1}$$

$$I^{1-\theta}x(0) = a, \theta = \theta_1 + \theta_2 - \theta_1\theta_2, \tag{2}$$

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where $D^{\theta_1,\theta_2}(\theta_1,\theta_2 \in \mathbb{C})$ is Hilfer fractional derivative of order $\theta_1 = \alpha+i\beta$ and of type $\theta_2 = \gamma+i\eta$. Here $0 < \theta_1 < 1$ and $0 \le \theta_2 \le 1$, with α, β, γ and $\eta \in R$. Let $f : J \times R \to R$ be a continuous function. The paper is organised as follows. In Section 2, we give some basic definitions and results concerning with the Hilfer fractional derivative. In Section 3, we present our main result by using fixed point theorem. In Section 4, we discuss four king of Ulam type stability.

2. Preliminaries

For the ease of the readers, we present some basic definitions and lemmas. Next, consider the following spaces. Let $C(J)$ be the Banach space of all continuous functions defined on J into R with the norm

$$\|x\|_C := \max\{|x(t)| : t \in J\}.$$

The weighted space $C_{1-\xi}(J)$ of functions f on J is defined by

$$C_{1-\xi}(J) = \left\{f : J \to R : t^{1-\xi}f(t) \in C(J)\right\}, 0 \le \xi (= \Re(\theta)) < 1,$$

with the norm

$$\|f\|_{C_{1-\xi}} = \left\|t^{1-\xi}f(t)\right\|_{C(J)} = \max_{t\in J}\left|t^{1-\xi}f(t)\right|.$$

An extensive on complex order, one can refer to [11, 12, 19].

Definition 2.1. *The Riemann fractional integral of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of a function f is defined by,*

$$I^\theta f(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}f(s)ds, \quad t \ge 0.$$

Definition 2.2. *The Riemann fractional derivative of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of a function f is defined by,*

$$D^\theta f(t) = \frac{1}{\Gamma(n-\theta)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\theta-1}f(s)ds, \quad t \ge 0.$$

where $n = [\Re(\theta)] + 1$.

Definition 2.3. *The Caputo fractional derivative of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of function f is defined by,*

$$D^\theta f(t) = \frac{1}{\Gamma(n-\theta)}\int_0^t (t-s)^{n-\theta-1}f^n(s)ds.$$

Definition 2.4. *The Hilfer fractional derivative of order $0 < \theta_1 < 1$ and $0 \le \theta_2 \le 1$ of function $f(t)$ is defined by*

$$D^{\theta_1,\theta_2}f(t) = (I^{\theta_2(1-\theta_1)}D(I^{(1-\theta_2)(1-\theta_1)}f))(t).$$

The Hilfer fractional derivative is used as an interpolator between the Riemann-Liouville and Caputo derivative.

Remark 2.1. (a) *The operator D^{θ_1,θ_2} also can be written as*

$$D^{\theta_1,\theta_2} = (I^{\theta_2(1-\theta_1)}D(I^{(1-\theta_2)(1-\theta_1)})) = I^{\theta_2(1-\theta_1)}D^\theta, \quad \theta = \theta_1 + \theta_2 - \theta_1\theta_2.$$

(b) *If $\theta_2 = 0 (\gamma = 0,\ \eta = 0)$, then $D^{\theta_1,\theta_2} = D^{\theta_1,0}$ is called Riemann-Liouville fractional derivative.*

(c) *If $\theta_2 = 1 (\gamma = 1,\ \eta = 0)$, then $D^{\theta_1,\theta_2} = I^{1-\theta_1}D$ is called Caputo fractional derivative.*

Definition 2.5. *The Stirling asymptotic formula of Gamma function for $z \in \mathbb{C}$ is following*

$$\Gamma(z) = (2\pi)^{1/2} z^{z-\frac{1}{2}} e^{-z} \left[1 + O\left(\frac{1}{z}\right)\right] \quad (|\arg(z)| < \pi; |z| \to \infty),$$

and its result for $|\Gamma(a+ib)|$, $(a, b \in R)$ is

$$|\Gamma(a+ib)| = (2\pi)^{1/2} |b|^{a-\frac{1}{2}} e^{-a-\frac{\pi|b|}{2}} \left[1 + O\left(\frac{1}{z}\right)\right] \quad (b \to \infty).$$

Here we adopt some definitions from [5, 19].

Definition 2.6. *Eq.(1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\xi}(J)$ of the inequality*

$$\left|D^{\theta_1,\theta_2} z(t) - f(t, z(t)\right| \leq \epsilon, \quad t \in J, \tag{3}$$

there exists a solution $x \in C_{1-\xi}(J)$ in Eq.(1) with

$$|z(t) - x(t)| \leq C_f \epsilon, \quad t \in J.$$

Definition 2.7. *Eq. (1) is generalized Ulam-Hyers stable if there exist $\psi \in C_{1-\xi}(J)$, $\psi_f(0) = 0$ such that for each solution $z \in C_{1-\xi}(J)$ of the inequality*

$$\left|D^{\theta_1,\theta_2} z(t) - f(t, z(t))\right| \leq \epsilon, \quad t \in J, \tag{4}$$

there exists a solution $x \in C_{1-\xi}(J)$ in Eq. (1) with

$$|z(t) - y(t)| \leq \psi_f \epsilon, \quad t \in J.$$

Definition 2.8. *Eq.(1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\xi}(J)$ if there exists a real number $C_f > 0$ such that for each solution $z \in C_{1-\xi}(J)$ of the inequality*

$$\left|D^{\theta_1,\theta_2} z(t) - f(t, z(t))\right| \leq \epsilon\varphi(t), \quad t \in J, \tag{5}$$

there exists a solution $x \in C_{1-\xi}(J)$ in Eq.(1) with

$$|z(t) - x(t)| \leq C_f \epsilon\varphi(t), \quad t \in J.$$

Definition 2.9. *Eq.(1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\xi}(J)$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C_{1-\xi}(J)$ of the inequality*

$$\left|D^{\theta_1,\theta_2} z(t) - f(t, z(t))\right| \leq \varphi(t), \quad t \in J, \tag{6}$$

there exists a solution $x \in C_{1-\xi}(J)$ in Eq.(1) with

$$|z(t) - x(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in J.$$

We state the following generalization of Gronwall's lemma for singular kernals.

Lemma 2.1. *Let $v : [0, T] \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, T]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^\alpha} ds.$$

Then there exists a constant $k = k(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_0^t \frac{W(s)}{(t-s)^\alpha} ds,$$

for every $t \in [0, T]$.

Theorem 2.1. *(Schauder fixed point theorem) Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E and $N : Q \to Q$ is compact, and continuous map. Then N has at least one fixed point in Q.*

Lemma 2.2. *If x is the solution of the equation Eq.(1)-(2), if and only if it satisfies the following integral equation*

$$x(t) = x_0 \frac{t^{\theta-1}}{\Gamma(\theta)} + \frac{1}{\Gamma(\theta_1)} \int_0^t (t-s)^{\theta_1 - 1} f(s, x(s))ds, \tag{7}$$

3. Existence and uniqueness results

In this section, we study the existence and uniqueness of solutions for Eq.(1)-(2). To treat this system, we introduce the following hypotheses.

(H1) Let $f : J \times R \to R$ be continuous. For $x, y \in R$, there exists a positive constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|.$$

(H2) There exists a constant m, n such that

$$|f(t,x)| \leq m + n|x|, \ \forall\, t \in J, \ x \in R.$$

(H3)

$$\rho = \frac{LB(\xi, \alpha)}{|\Gamma(\theta_1)|} T^{\alpha}.$$

(H4) Suppose that there exists $\lambda_\varphi > 0$ such that

$$I^{\theta_1}\varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 3.1. *Assume that [H1] and [H2] are satisfied. Then, Eq.(1)-(2) has at least one solution.*

Proof. Consider the operator $N : C_{1-\xi}(J) \to C_{1-\xi}(J)$ given by

$$(Nx)(t) = x_0 \frac{t^{\theta-1}}{\Gamma(\theta)} + \frac{1}{\Gamma(\theta_1)} \int_0^t (t-s)^{\theta_1-1} f(s, x(s))ds. \tag{8}$$

It is obvious that the operator N is well defined. Clearly, the fixed points of the operator N are solutions of the problem. For any $x \in C_{1-\xi}(J)$ and each $t \in J$ we have,

$$\begin{aligned}
|(Nx)(t)| &\leq x_0 \frac{|t^{\theta-1}|}{|\Gamma(\theta)|} + \frac{1}{|\Gamma(\theta_1)|} \int_0^t \left|(t-s)^{\theta_1-1}\right| |f(s,x(s))|\, ds \\
&\leq x_0 \frac{t^{\xi-1}}{|\Gamma(\theta)|} + \frac{1}{|\Gamma(\theta_1)|} \int_0^t (t-s)^{\alpha-1} |m + nx(s)|\, ds, \\
\|(Nx)(t)\|_{C_{1-\xi}} &\leq \frac{x_0}{|\Gamma(\theta)|} + \frac{t^{1-\xi}}{|\Gamma(\theta_1)|} m \int_0^t (t-s)^{\alpha-1} ds + \frac{t^{1-\xi}}{|\Gamma(\theta_1)|} n \int_0^t (t-s)^{\alpha-1} |x(s)|\, ds \\
&\leq \frac{x_0}{|\Gamma(\theta)|} + \frac{t^{1-\xi}}{\alpha|\Gamma(\theta_1)|} m t^{\alpha} + \frac{t^{1-\xi}}{|\Gamma(\theta_1)|} n B(\xi,\alpha) t^{\alpha+\xi-1} \|x\|_{C_{1-\xi}} \\
&\leq \frac{x_0}{|\Gamma(\theta)|} + \frac{m}{\alpha|\Gamma(\theta_1)|} T^{\alpha+1-\xi} + \frac{nB(\xi,\alpha)}{|\Gamma(\theta_1)|} T^{\alpha} \|x\|_{C_{1-\xi}} := r.
\end{aligned}$$

This proves that N transforms the ball $B_r = \left\{x \in C_{1-\xi}(J) : \|x\|_{C_{1-\xi}} \leq r\right\}$ into itself. The proof is divided into several steps. $\square$

Step 1: Now we show that $N : B_r \to B_r$ is continuous.
Let x_n be a sequence such that $x_n \to x$ in B_r. Then for each $t \in J$, we have

$$\begin{aligned}
&\left|((Nx_n)(t) - (Nx)(t))\, t^{1-\xi}\right| \\
&\leq \left|\frac{t^{1-\xi}}{\Gamma(\theta_1)} \int_0^t (t-s)^{\theta_1 - 1} f(s, x_n(s))ds - \frac{t^{1-\xi}}{\Gamma(\theta_1)} \int_0^t (t-s)^{\theta_1 - 1} f(s, x(s))ds\right| \\
&\leq \frac{t^{1-\xi}}{|\Gamma(\theta_1)|} \int_0^t (t-s)^{\alpha-1} \left|f(s, x_n(s)) - f(s, x(s))\right| ds \\
&\leq \frac{t^{1-\xi}}{|\Gamma(\theta_1)|} B(\xi, \alpha)(t-s)^{\alpha+\xi-1} \left\|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\right\|_{C_{1-\xi}}.
\end{aligned}$$

Since f is continuous, then by the Lebesgue dominated convergence theorem which implies

$$\|(Nx_n) - (Nx)\|_{C_{1-\xi}} \to 0 \text{ as } n \to \infty.$$

Step 2: $N(B_r)$ is uniformly bounded.
It is clear that $N(B_r) \subset B_r$ is bounded.
Step 3: $N(B_r)$ is equicontinuous.
Let $t_1, t_2 \in J, t_1 < t_2$. Then,

$$\begin{aligned}
&\left|(Nx)(t_1)t_1^{1-\xi} - (Nx)(t_2)t_2^{1-\xi}\right| \\
&\leq \frac{t_1^{1-\xi}}{|\Gamma(\theta_1)|} \int_0^{t_1} (t_1 - s)^{\alpha-1} |f(s, x(s))|\, ds - \frac{t_2^{1-\xi}}{|\Gamma(\theta_1)|} \int_0^{t_2} (t_2 - s)^{\alpha-1} |f(s, x(s))|\, ds \\
&\leq \frac{\|f\|_{C_{1-\xi}}}{|\Gamma(\theta_1)|} B(\xi, \alpha) \left|t_1^\alpha - t_2^\alpha\right|.
\end{aligned}$$

As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that N is continuous and compact. From an application of Schauder's Theorem 2.1, we deduce that N has a fixed point x which is a solution of the problem Eq. (1)-(2).

Theorem 3.2. *If hypothesis (H1) and (H3) are fulfilled. Then the Eq.* (1)-(2) *has a unique solution.*

Proof. By Eq. (7), it is clear that the fixed points of N is solutions of Eq. (1).
Let $x, y \in C_{1-\xi}(J)$ and for $t \in J$, we have

$$\left|((Nx)(t) - (Ny)(t))\, t^{1-\xi}\right| \leq \frac{t^{1-\xi}}{\Gamma(\theta_1)} \int_0^t \left|(t-s)^{\theta_1-1}\right| \left|(f(s, x(s)) - f(s, y(s)))\right| ds$$

$$\leq \frac{t^{1-\xi}}{|\Gamma(\theta_1)|} \int_0^t (t-s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))|\, ds \leq \frac{Lt^{1-\xi}}{|\Gamma(\theta_1)|} B(\xi, \alpha) t^{\alpha+\xi-1} \|x - y\|_{C_{1-\xi}}$$

$$\leq \frac{LB(\xi, \alpha)}{|\Gamma(\theta_1)|} T^\alpha \|x - y\|_{C_{1-\xi}} := \rho \|x - y\|_{C_{1-\xi}}.$$

$\square$

4. Stability results

Next, the criteria of Ulam-Hyers stability and generalised Ulam-Hyers-Rassias stability for differential equations under complex order Hilfer fractional derivative is analysed.

Remark 4.1. *A function $z \in C_{1-\xi}(J)$ is a solution of the inequality*

$$\left|D^{\theta_1,\theta_2}z(t) - f(t,z(t))\right| \le \epsilon, \quad t \in J,$$

if and only if there exist a function $g \in C_{1-\xi}(J)$ such that

(i) $|g(t)| \le \epsilon, t \in J$.
(ii) $D^{\theta_1,\theta_2}z(t) = f(t,z(t)) + g(t), t \in J$.

Remark 4.2. *If z is solution of the inequality (3), then z is a solution of the following integral inequality*

$$\left|z(t) - z_0\frac{t^{\theta-1}}{\Gamma(\theta)} - \frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1} f(s,z(s))ds\right| \le \left(\frac{T^\alpha}{\alpha\,|\Gamma(\theta_1)|}\right)\epsilon.$$

Indeed, by Remark 4.1 we have that

$$D^{\theta_1,\theta_2}z(t) = f(t,z(t)) + g(t), \; t \in J.$$

Then

$$z(t) = z_0\frac{t^{\theta-1}}{\Gamma(\theta)} + \frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1}(f(s,z(s)) + g(s))ds.$$

From this it follows that

$$\left|z(t) - z_0\frac{t^{\theta-1}}{\Gamma(\theta)} - \frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1} f(s,z(s))ds\right| \le \left|\frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1} g(s)ds\right|$$
$$\le \frac{1}{|\Gamma(\theta_1)|}\int_0^t \left|(t-s)^{\theta_1-1}\right| |g(s)|\, ds \le \frac{1}{|\Gamma(\theta_1)|}\int_0^t \left|(t-s)^{\alpha-1}\right| |g(s)|\, ds \le \left(\frac{T^\alpha}{\alpha\,|\Gamma(\theta_1)|}\right)\epsilon.$$

We have similar remarks for the inequality (4), (5) and (6).
Now, we give the main results, generalised Ulam-Hyers-Rassias stable results.

Theorem 4.1. *The hypotheses [H1] and [H4] holds. Then Eq.(1)-(2) is generalised Ulam-Hyers-Rassias stable.*

Proof. Let z be solution of inequality 6 and by Theorem 3.2 there x is a unique solution of the problem

$$D^{\theta_1,\theta_2}x(t) = f(t,x(t)),$$
$$I^{1-\theta}x(0) = I^{1-\theta}z(0) = x_0, \quad \theta = \theta_1 + \theta_2 - \theta_1\theta_2.$$

Then we have

$$x(t) = x_0\frac{t^{\theta-1}}{\Gamma(\theta)} + \frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1} f(s,x(s))ds.$$

By differentiating inequality (6), we have

$$\left|z(t) - x_0\frac{t^{\theta-1}}{\Gamma(\theta)} - \frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1} f(s,z(s))ds\right| \le \left|\frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1}\varphi(s)ds\right| \le \lambda_\varphi\varphi(t).$$

Hence it follows that,

$$\begin{aligned}|z(t)-x(t)| &\leq \left|z(t)-x_0\frac{t^{\theta-1}}{\Gamma(\theta)}-\frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1}f(s,x(s))ds\right| \\ &\leq \left|z(t)-x_0\frac{t^{\theta-1}}{\Gamma(\theta)}-\frac{1}{\Gamma(\theta_1)}\int_0^t (t-s)^{\theta_1-1}f(s,z(s))ds\right| \\ &+\left|\int_0^t (t-s)^{\theta_1-1}\left(f(s,z(s))-f(s,x(s))\right)ds\right| \leq \lambda_\varphi\varphi(t)+\frac{LT^\alpha}{\alpha\left|\Gamma(\theta_1)\right|}\left|z-x\right|.\end{aligned}$$

By Lemma 2.1., there exists a constant $M^*>0$ independent of $\lambda_\varphi\varphi(t)$ such that

$$|z(t)-x(t)| \leq M^*\varphi(t).$$

Thus, Eq.(1)-(2) is generalized Ulam-Hyers-Rassias stable. □

5. Conclusion

In this research work we have considered a class of fractional differential equations involving Hilfer fractional derivative of complex order. We have investigated existence theory as well as various kinds of Ulam stability results for the solutions of the considered problem.

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